



# Air Traffic Management

## teacher guide

### Activity I: Determining the Locus of a Flight Plan

#### lesson overview

**Materials:** graph paper, protractor, and ruler

**Time for set-up:** none

**Time for lesson:** Part A = 30 minutes, Part B = 1 hour, Part C = 45 minutes, Part D = 45 minutes, Part E = 45 minutes, Part F = 45 minutes, Part G = 30 minutes, Part H = 1 hour

#### Student Prerequisites:

Part A: PRE, some knowledge of  $d = rt$  and substitution, ability to graph points onto coordinate graphs.

Part B: PRE, GEO, good mathematic reasoning skills, understanding of proof organization, Pythagorean Theorem, basic trigonometry, graphing points / use of coordinate geometry and vectors

Part C: A1, GEO, ability to write and manipulate equations of lines, draw linear graphs, understanding of basic proof organization

Part D: GEO, understanding of basic proof organization, understanding of basic CALC, ability to use Pythagorean Theorem and distance formula, basic TRIG

Part E: basic TRIG, ability to complete a 3-column proof

Part F: GEO, understanding of basic proof organization, ALG (including basic radicals)

Part G: A1 (radicals, solving formulas)

Part H: A1 (linear, parabolic equations), GEO (coordinate geometry, circular equations, vectors)

#### Icons for recommended subject areas where activities could be used:

Part A: A1

Part B: GEO, A2, TRIG, CALC

Part C: A1, GEO

Part D: GEO, CALC

Part E: TRIG

Part F: A1, GEO (as intro to vectors), SS (geography, map use)

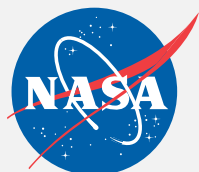
Part G: A1

Part H: GEO, A2

#### Objectives

Part A: students are able to manipulate algebraic equations, to come up with generalized equations for use in evaluating a number of situations. Students review use of  $d = rt$ , students practice using basic algebra, including multiplication and addition of negative numbers.

Part B: students are able to set up a column-form proof, using algebraic properties and some basic trigonometry, proof using very simple calculus. Students are able to identify possible ways to combine vectors, using Pythagorean Theorem, trigonometry. Students can combine vectors using these techniques. Students can illustrate examples graphically, using coordinate geometry. and use estimation techniques and/or





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the distance formula to check other methods/solutions. Students are able to identify special circumstances with negative numbers and create generalizations about dealing with these solutions.

Part C: students able to complete a 3-column proof, students able to pictorially and literally identify a locus, students able to evaluate (compare and contrast) use of one equation versus another, students able to determine if points occur in lines or not, students able to solve for unknown variables in complex multi-variable equations, students able to use evaluation to determine "truth" of equations, students able to graph lines and vectors

Part D: students able to complete a proof with calculus, students able to compare 2 or more methods doing same thing (geo to calculus) in order to check work and evaluate processes, students able to use Pythagorean Theorem, students able to use basic Theorems / Postulates in Geometry to justify a procedure, students able to use basic trigonometry, students able to use distance formula, students able to use a combination of evaluation and algebra skills to solve an equation.

Part E: students able to complete a proof using trigonometry, students able to use trig, radicals (and their properties), basic algebra to solve equations, students able to analyze results in order to generalize patterns, predict outcomes, and find possible sources of misunderstanding, students able to draw figures with certain angles, dimensions, vectors, students will practice using negative numbers, rounding decimals

Part F: students will complete a proof with basic algebra properties, students will solve equations using evaluation and algebra, students will gain a basic understanding of components of vectors, students will use geographical maps to gain a sense of latitude, longitude, distance, and direction, students will link math to real-life situations

Part G: students will logically derive an equation, students will use evaluation and algebra to solve for unknowns, students will use radicals (and associated properties)

Part H: students write and use equations for circular, parabolic, and linear structures, students use coordinate geometry, students describe a complex path using vectors

### Student Assessments

Part A, C, and E: Locus Student Worksheet, some graphs.

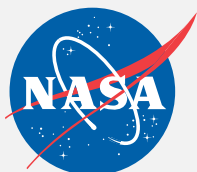
Part B Locus Student Worksheet, graphs / illustrations, prose about special solutions (negative numbers)

Part D: Locus Student Worksheet, some work and graphs on other paper

Part F: Locus Student Worksheet, some prose / discussion, some graphs

Part G: Locus Student Worksheet

Part H: Locus Student Worksheet, several sheets showing calculations, approximations





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#### Introduction

As soon as a flight path is decided upon, the pilot and controllers monitor the course of the airplane to be sure the airplane stays on that course. This is not unlike making sure a car stays on a road (and between the lines). If we describe a flight path in terms of a locus defined by lines with equations, it should be easy to determine if we are inside the locus and if the actual airplane's path intersects any of the locus' boundaries (or lines). Controllers have programs that do essentially this, and they can foresee when an airplane may leave this locus. To better understand how such programs work, let's look at a flight path specifically in terms of objective points, then regions (paths).

This is broken up into eight sections:

**Part A. At what point will the airplane be after time,  $t$ ?**

**Part B. What is the distance traveled in time  $t$ ?**

**Part C. Can we reach a specific endpoint if given  $v_x$  and  $v_y$  are constants?**

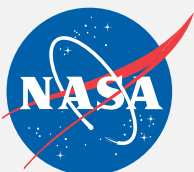
**Part D. If the airplane does not reach  $(x_1, y_1)$ , what is the closest point of approach  $(x_2, y_2)$  where it should turn to get to  $(x_1, y_1)$ ?**

**Part E. If we want to reach  $(x_1, y_1)$ , what should the heading angle ( $x$ ) be?**

**Part F. If I want to reach  $(x_1, y_1)$  at  $t = t_f$ , what should I do?**

**Part G. If the airplane flies for a certain speed for a certain amount of time, at what speed should the airplane fly for the remaining portion of time, in order to still reach its destination at time  $t_f$ ?**

**Part H. Staying on the geometric path**



## Activity I: Determining the Locus of a Flight Plan

### Part A - Where will the airplane be after time t?

1. What I already know:

a)  $d = \underline{\text{r}} \times \underline{\text{t}}$ , where

$d$  = distance,  $\underline{\text{r}}$  = rate, and  $\underline{\text{t}}$  = time

This means, if I start from the origin (0,0) and travel at 40 mi/h for 2 hours, I will have traveled 80 miles.

If I then travel for another hour at 20 mi/h, I will have travelled another 20 miles.

The total trip will be 100 miles because total distance traveled = sum of all parts of trip.

The teacher may want to review these principles with students before continuing. The teacher could elaborate the last sentence on the student worksheets, asking what could replace one of the  $d_1$  or  $d_2$  in the following equation:  $d_{\text{total}} = d_1 + d_2$ . Hopefully students will see that substitution is easily applied.  $d_{\text{total}} = d_1 + r_2 t_2$  OR  $d_{\text{total}} = r_1 t_1 + r_2 t_2$ .

b) Not all distances are known or can be easily calculated for all problems.

Sometimes, we have to leave things in variable form. If you were to write an equation with variables to show that the total distance = the sum of all distances, what would the equation look like? Use  $d$  and subscripts to denote different distances.

$$d_{\text{total}} = d_1 + d_2 + d_3 + \dots d_n$$

$$d_{\text{total}} = \text{summation symbol for } x = 1 \text{ to } n \text{ of } (d_x)$$

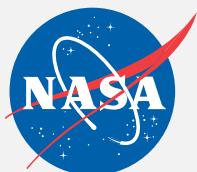
If you don't know the distance of the first stretch of the path, and only want to use the  $r$  and  $t$  variables for that path, what would the new equation look like?

$$d_{\text{total}} = r_1 t_1 + d_2 + d_3 + \dots d_n$$

What is another way of writing this equation, using no  $d$ s?

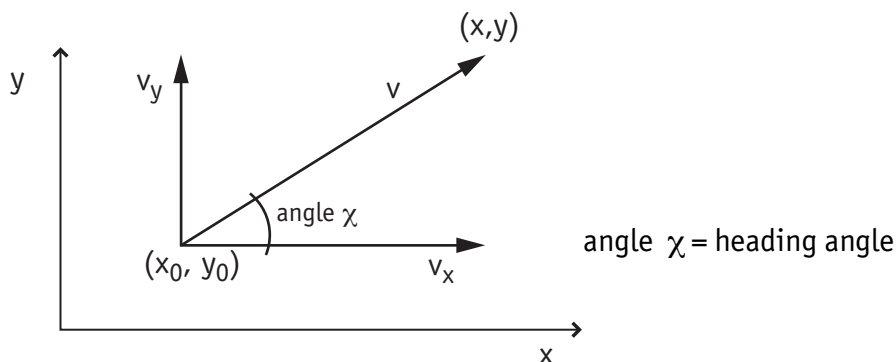
$$d_{\text{total}} = r_1 t_1 + r_2 t_2 + r_3 t_3 + \dots r_n t_n$$

$$d_{\text{total}} = \text{summation symbol for } x = 1 \text{ to } n \text{ of } (r_x t_x)$$



### Activity I: Determining the Locus of a Flight Plan

- c) Our problem becomes a little more complex when we use  $x$  and  $y$  to illustrate points and we travel along diagonals. We can use the following variables to determine where an airplane will be  $(x, y)$  after time  $t$ .



Similarly as with distance, the final position = where the pilot began from  $(x_0, y_0)$  plus the new distance traveled. For convenience, the distance vector is broken up into its  $x$  and  $y$  constituents.

$$x = x_0 + v_x t$$

$$y = y_0 + v_y t$$

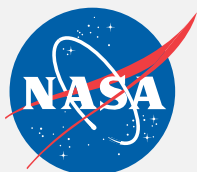
Have students discuss the variables and picture in order to assess comprehension. Many students will be confused about the path the airplane is traveling on, and on where the  $x$  and  $y$  vectors came from. You might also discuss why  $t$  in the two equations is the same, while  $v$  is different.

2. Solve for the unknown variables; fill in the chart.

	$x_0$	$y_0$	$v_x$	$v_y$	$t$	$x$	$y$
1.	0	0	3	2	5	15	10
2.	2	2	60	25	3	92	77
3.	4	1	-10	30	5	-45	151
4.	5	10	-15	-15	2	-25	-40
5.	-10	2	17	20	2	24	42

3. For any two of the above situations, summarize the path of the airplane as the example shows, on a coordinate grid. Estimate the distance traveled using a ruler or the squares on your graph paper, or use the distance formula if you know it.

Be sure students are careful! These will be used in the next section.





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#### Part B - What is the distance traveled in time $t$ ?

Let's derive a formula then use it to solve for distance with respect to time!

1. First method of derivation: Proof with Algebra and Trigonometry Properties.

Given: distance is defined with the following equation:

$$d = ((x - x_0)^2 + (y - y_0)^2)^{1/2}$$

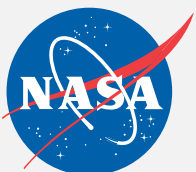
and

$$x = x_0 + v_x t$$

$$y = y_0 + v_y t \quad (\text{and variables defined in picture in Part A})$$

Prove:  $d = vt$

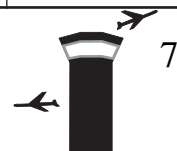
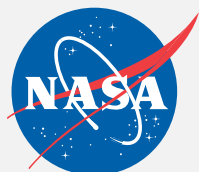
You can have students do this using only the above information, or you can have them fill in blanks, as in the following chart.



### Activity I: Determining the Locus of a Flight Plan

The following proof requires knowledge of algebra properties, so it's a good example for early Geometry. It also uses some trigonometry, so it would be appropriate for more advanced students as well.

	Statements	Reasons	Reference Numbers
1.	$x = x_0 + v_x t$ $y = y_0 + v_y t$	Given	-----
2.	$x - x_0 = v_x t$ $y - y_0 = v_y t$	Subtraction Property of Equality	(1)
3.	$d = ((x - x_0)^2 + (y - y_0)^2)^{1/2}$	Given	-----
4.	$d = ((v_x t)^2 + (v_y t)^2)^{1/2}$	Substitution Property	(2,3)
5.	$d = t (v_x^2 + v_y^2)^{1/2}$	Distributive Property	(4)
6.	$\cos \chi = v_x / v$ $\sin \chi = v_y / v$	Definition of cosine and sine, given figure	fig or -----
7.	$v_x = v \cos \chi$ and $v_y = v \sin \chi$	Multiplication Property of Equality	(6)
8.	$v_x + v_y = v \cos \chi + v \sin \chi$	Addition Property of Equality	(7)
9.	$v_x + v_y = (v \cos \chi + v \sin \chi)$	Distributive Property	(8)
10.	$v_x^2 + v_y^2 = v^2(\cos^2 \chi + \sin^2 \chi)$	Multiplication Property of Equality	(9)
11.	$v_x^2 + v_y^2 = v^2(1)$	$\cos^2 \chi + \sin^2 \chi = 1$ Pythagorean Identity	(10)
12.	$d = t (v^2)^{1/2}$	Substitution Property	(5, 11)
13.	$d = tv$	Definition of Square Root	(12)



### Activity I: Determining the Locus of a Flight Plan

2. Use calculus to derive the same formula.

For calculus students, I recommend going through the proof first, as most students will find it somewhat tedious. Following the proof with calculus should show students how calculus can be a real time-saver.

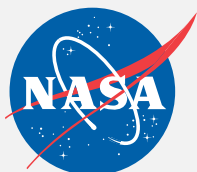
$\delta$  is delta, or change.

Given: Definition of velocity = rate of change in position or change in distance over change in time:  $\delta d / \delta t = v$ ,

Prove:  $d = vt$

	Statements	Reason/Explanation	Reference Numbers
1.	$\delta d / \delta t = v$	given	-----
2.	$\delta d = v \delta t$	multiplication property of equality	(1)
3.	$\delta d = v dt$	take integral of both sides	(2)
4.	$d = vt + c$	solving, definition of integral	(3)
5.	$d = vt$	at $t = 0$ and $d = 0$ (initial conditions)	(4)

The math here is very simple - so simple in fact that I think it is worthy of using in a class well before calculus, to help build in students an appreciation for advanced math (and a preview for what is to come). You could explain this short proof by explaining that the  $\delta$ s represent deltas, or change. Therefore, step 1 is a perfect translation of the given definition. All students should be able to follow step 2. In step 3, the  $\int$  represents the integral, which is essentially the same as a summation symbol or a symbol that indicates that a series of additions must take place if the domain is a specific range. (Even prealgebra kids should be able to make sense of this, as evaluation of the same expression for a series of numbers, then adding the numbers together.) If we are starting at a point in space, then moving a bit and stopping, then moving a bit and stopping, we will be measuring each change in position (or each distance), then adding all distances together.

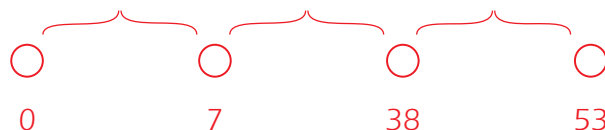


### Activity I: Determining the Locus of a Flight Plan

For instance, you could talk about travel from mile marker to mile marker:

First you could solve for  $\delta d$ :

$$\delta d = 7 - 0 = 7 \quad 38 - 7 = 31 \quad 53 - 38 = 15$$

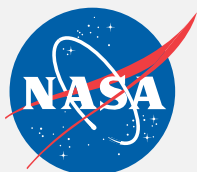


Then for  $\int \delta d = 7 + 31 + 15 = 53$

Therefore, as with  $\int \delta d$ , we can use integrals to find complete distances! (And calculus is largely about seeing WHOLE PICTURES rather than pieces.)

In step 4, then, students should be able to see that the integral of several changes in position is the same thing as one distance, and integrals of several changes in time is the same thing as one time. Since velocity does not differ according to our model, taking an integral of it does not change it. Because we do not know where the object that is moving started, we must include a constant to represent this information. The constant itself will not effect our velocity, which is why it was not included it before. The constant reflects original distance or time x velocity.

In step 5, we assume that the initial distance was zero (started at origin) when time was zero, so that the constant equals zero. This allows us to create the simplistic equation  $d = vt$ .



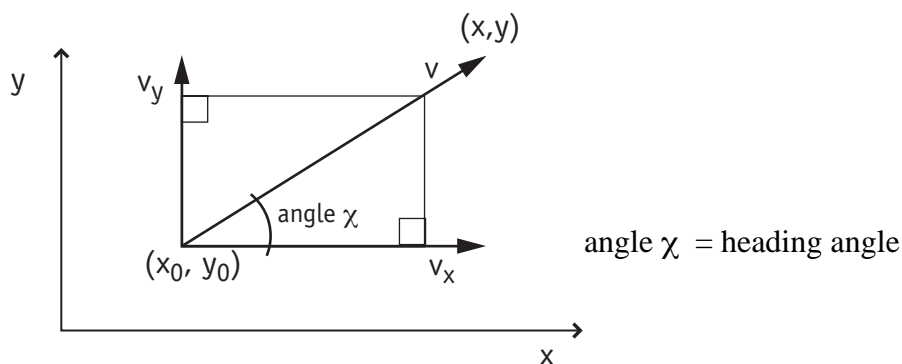
### Activity I: Determining the Locus of a Flight Plan

3. We can solve for distance using  $d = vt$ .

To do this, we must use a single vector, as opposed to the vector components,  $v_x$ , and  $v_y$ .

a) How could you use the vector components to find the length of  $v$ ?

The following drawing should help you.:



Some will embrace the formulas in this text, while most will look at the picture and see a geometric relationship between the vectors. Some may see in this relationship a quadrilateral, specifically a square or rectangle, with right angles that can be used with Pythagorean Theorem to solve for the diagonal ( $v$ ). Others will attempt to use the heading angle and some trigonometry to find the diagonal ( $v$ ). These are summarized below.

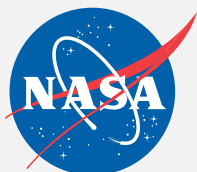
The single vector can be determined using Pythagorean Theorem, with the legs of the triangle as  $v_x$  and  $v_y$ , and the hypotenuse as  $v$ .

$$v = (v_x^2 + v_y^2)^{1/2}.$$

The single vector can be determined using trigonometry and the heading angle.

$$\sin \chi = v_y/v \quad \text{so} \quad v = v_y/\sin \chi.$$

Students might also obtain equations from the previous proofs that are similar to or the same as these.

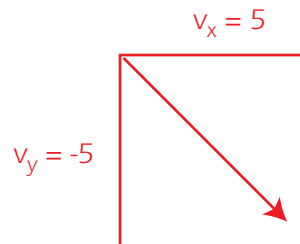


### Activity I: Determining the Locus of a Flight Plan

b) Vectors can be negative, as well as positive.

i) If a vector was negative, what would that mean? Draw an example of a situation where a vector is negative.

example: moving down and to the right, like with  $v_x = 5$ ,  $v_y = -5$ .



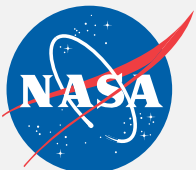
ii) Which of the above methods of solving provides no negative vectors.

Explain why no negative vectors are included in the solution for v.

Using Pythagorean Theorem to solve would yield only positive vectors because vector components are squared to find distance. Even if a vector component is negative,  $(\text{negative})^2$  yields a positive.

iii) When would the other method provide a negative v vector?

When  $v_x$  and  $v_y$  are negative, a negative vector would be produced.



### Activity I: Determining the Locus of a Flight Plan

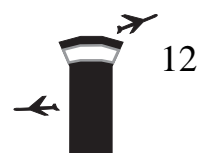
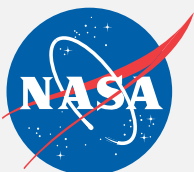
c) Solve for the unknown in the following chart, using Pythagorean Theorem and  $d = vt$ .

	$v_x$	$v_y$	$v$	$t$	$d$
1.	5	3	$(34)^{1/2}$	5	$5(34)^{1/2}$
2.	4	7	$(65)^{1/2}$	12	$(9360)^{1/2}$
3.	$3(5)^{1/2}$	6	9	3.5	31.5
4.	$3(7)^{1/2}$	9	12	7.5	90

Draw any two situations from the chart on graph paper, using coordinate geometry, and estimate the variables by using a ruler and/or the distance formula. Check your answers against the estimates - do they make sense?

d) Solve for the unknowns, using trigonometry and  $d = vt$ .

	Heading Angle (degrees)	$v_x$ or $v_y$	$v$	$t$	$d$
1.	19°	$v_x = 12$	$12/\cos(19) = 12.6914482$	3	38.07434452
2.	78°	$v_y = 20$	$20/\sin(78) = 20.4468119$	4	81.78724759
3.	45°	$v_x = 27 \cos(45)$ $= 19.091883$	27	6	162
4.	30°	$v_y = 30 \sin(30)$ $= 15$	30	2	60
5.	-15°	$v_x = -25 \cos(-15)$ $= -24.1481456$	-25	6	-150





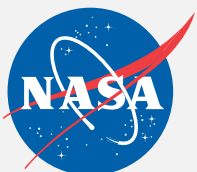
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Draw any two of the situations from question 3 on graph paper with a protractor, and the variables using a ruler. Check your answers against the estimates - do they make sense? What is an important thing about negative values in these problems that must be realized so you do not erroneously say there is “no solution” ?

The negative sign on heading angles and vectors has more to do with direction than size. The negative distance in example 5 reflects the fact that a negative vector was used; distance was lost in this leg of the journey.



### Activity I: Determining the Locus of a Flight Plan

**Part C - Can we reach a specific endpoint if given  $v_x$  and  $v_y$  are constants?**

In other words, will  $(x_1, y_1)$  be a point in our path, if  $v_x$  and  $v_y$  are constants?

Ask students to explain this scenario, first. A simple explanation might be that you can only go a limited speed in one direction (thus each directional vector is limited). It is easy to understand how speed might be limited - many things can only go so fast, we might use the cruise control on a car, for instance if we choose to travel at the most energy-efficient speed. It is a little more difficult to think of why we might only be able to go in one direction - there are obstacles so we can't go any other direction, we must stay on our flight path - but all students should know that the shortest distance between two points is a straight line. Discuss this scenario thoroughly, so students have a sense of owning the problem.

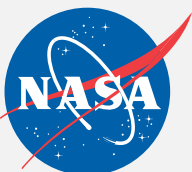
1. Draw the two possible situations here :  $(x_1, y_1)$  is in the locus of the path, or it is not.



( answers will vary)

2. How could you distinguish algebraically between the two options above?

If  $(x_1, y_1)$  is part of the path (line), or works in the line's equation, then I know that  $(x_1, y_1)$  is in the path.





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### Activity I: Determining the Locus of a Flight Plan

3. Let's write an equation for a line, using a) what we know about lines, and  
b) what we know already about the relationships between variables.

What is the equation of a line, which is easy to plug possible points into? Give the general form of the equation and then show an example of plugging a point into it.

Objective of this is for students to recall formulas, understand which are useful for testing point occurrence in lines, show examples of testing. This is similar to a "prediction" portion of a science report, where students foresee what the results will look like if their hypotheses are correct or incorrect.

**General Form:**  $y = mx + b$   
(slope-intercept form)

**Example:**  $y = 2x + 3$  and point (1, 2)

$$2 = 2(1) + 3$$

$$2 = 2 + 3$$

$$2 = 5$$

Not True! Point is not found on the line.

**General Form:**  $y_1 - y_0 = m(x_1 - x_0)$   
(point-slope form)

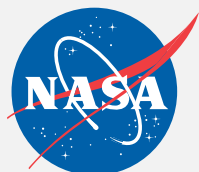
**Example:**  $y_1 - 3 = 7(x_1 - 4)$  and point (4, 3)

$$3 - 3 = 7(4 - 4)$$

$$0 = 7(0)$$

$$0 = 0$$

True! Point is found on the line.



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4. Let's come up with a formula that uses facts we already know. That is, given:

$$x = x_0 + v_x t \text{ and } y = y_0 + v_y t,$$

create an equation for a line that is in one of the above forms.

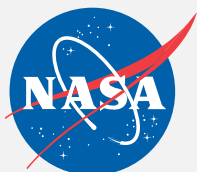
The proof that follows creates slope-intercept form.

Go through this process, below, and fill in the blanks.

Statements	Reasons	Reference No.s
1. $x = x_0 + v_x t$	Given	- - - - -
2. $x - x_0 = v_x t$	Subtraction Property of Equality	(1)
3. $(x - x_0) / v_x = t$	Division Property of Equality	(2)
4. $y = y_0 + v_y t$	Given	- - - - -
5. $y = y_0 + v_y ((x - x_0) / v_x)$	Substitution	(3, 4)
6. $y = y_0 + (v_y / v_x) x - (v_y / v_x) x_0$	Distributive Property	(5)
OR		
6. $y - y_0 = (v_y / v_x)(x - x_0)$	Subtraction Property of Equality (in Point-Slope form)	(5)
7. $y = (v_y / v_x)x + (y_0 - (v_y / v_x)x_0)$	Reorganization of Terms (in Slope-Intercept form)	(6)

5. Before you solve some equations, determine which situations would be ideal for using each equation.

- If given: **everything except  $v_y$**  it is best to use the Point-Slope Form equation.
- If given: **everything but  $y$**  it is best to use the Slope Intercept Form equation.
- If given: **everything but  $y_0$**  it is best to use the Point-Slope Form equation.
- If given: **everything but  $v_x$**  it is best to use the Point-Slope Form equation.
- If given: **everything but  $x$  or  $x_0$** , it is best to use the Point-Slope Form equation.

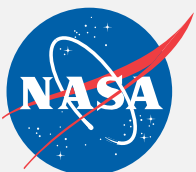


### Activity I: Determining the Locus of a Flight Plan

6. Will the following planes reach their destinations?

	$x_0$	$y_0$	$v_x$	$v_y$	$x$	$y$	Reach $(x_1, y_1)$ ? Yes/No
1.	2	5	6	13.8	12	28	yes
2.	1	-7	25	32	19	16.04	yes
3.	0	9	1	-2	8	-7	yes
4.	31	25	-5	-4	11	9	yes
5.	7	9	2	3	19	13	no
6.	14	0	5	-5	0	14	yes
7.	3	-5	-12	40	9	-25	yes
8.	7	-2.5	11	8	18	5	no
9.	-6	3.125	132	47.5	27	15	yes
10.	0.5	0.67	43	68	7.67	12	yes

7. Draw four of the above situations on graph paper. Use the vectors to draw the path from  $(x_0, y_0)$ . Be sure your drawing agrees with your answers above!



### Activity I: Determining the Locus of a Flight Plan

**Part D - If the airplane does not reach  $(x_1, y_1)$ , what is the closest point of approach  $(x_2, y_2)$  where it should turn to get to  $(x_1, y_1)$ ?**

1. Based on what you know from geometry, what is the measure of the angle for turning such that the path between the original destination  $(x_2, y_2)$  and  $(x_1, y_1)$  is shortest?

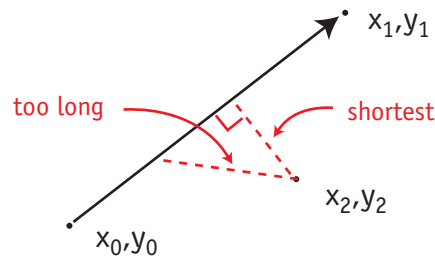
90 degrees

2. On graph paper, redraw one of the examples from Part C where the airplane did not arrive at  $(x_1, y_1)$ . Add  $(x_2, y_2)$  and the measure of the angle of turning in the drawing.

3. If  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$  serve as vertices, what kind of shape do you get (be specific)? Right Triangle

Need a hint for the answer above?

Answer this: In the graphic below, if you are travelling along the vector, where would you turn so you travel the shortest distance from the vector, to get to  $(x_2, y_2)$ ?



4. Solve for the following:

$(x_2, y_2) =$  \_\_\_\_\_

distance between  $(x_0, y_0)$  and  $(x_1, y_1) =$  \_\_\_\_\_

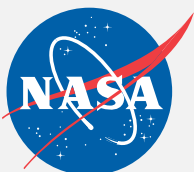
distance between  $(x_2, y_2)$  and  $(x_1, y_1) =$  \_\_\_\_\_

distance between  $(x_0, y_0)$  and  $(x_2, y_2) =$  \_\_\_\_\_

measures of other two angles = \_\_\_\_\_ degrees and \_\_\_\_\_ degrees

Answers vary depending on which row was chosen from chart.

Label these items on your drawing.



### Activity I: Determining the Locus of a Flight Plan

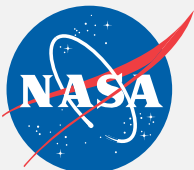
5. We can also find the closest point of approach using calculus. Use the partial proof below to explain how we derive a formula for this, using calculus.

	Statements	Reasons	Reference No.s
1.	$d_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$	given	- - - - -
2.	$\frac{\delta (d_2^2)}{(\delta x_2)} = 2(x_2 - x_1)(\delta x_2 / \delta x_2) + 2(y_2 - y_1)(\delta y_2 / \delta x_2)$	Derivative of Distance Formula	(1)
3.	$\delta (d_2^2) / (\delta x_2) = 0$	Condition for Extremum (because we are interested in the shortest distance between two points)	- - - - -
4.	$\delta y_2 / \delta x_2 = v_y / v_x$	Velocity Vectors	- - - - -
5.	$0 = x_2 - x_1 + (y_2 - y_1)(v_y / v_x)$	Substitution	(2, 3, 4)
6.	$y_2 = (v_y / v_x)x_2 + (y_0 - (v_y / v_x)x_0)$	Given. Equation for line between $(x_0, y_0)$ and $(x_2, y_2)$ . (See equation derived in Section C)	- - - - -
7.	$0 = x_2 - x_1 + (v_y / v_x)[(v_y / v_x)x_2 + (y_0 - (v_y / v_x)x_0)] - (v_y / v_x)y_1$	Substitution	(5, 6)
8.	$0 = x_2[1 + (v_y / v_x)^2] - x_1 + (v_y / v_x)(y_0 - v_y / v_x x_0) - (v_y / v_x)y_1$	(Solving) terms reordered, $x_2$ is factored	(7)
9.	$x_2 = \frac{x_1 + (v_y / v_x)y_1 - [v_y / v_x(y_0 - (v_y / v_x)x_0)]}{1 + (v_y / v_x)^2}$	Division Property of Equality	(8)

As soon as we have solved for  $x_2$ , we can substitute this value into our equation for our line containing  $(x_0, y_0)$  and  $(x_2, y_2)$ , to find  $y_2$ .

6. Use your values from questions 1 to 4 to solve for  $x_2$  and  $y_2$ .

Answers should agree with those from #4.



### Activity I: Determining the Locus of a Flight Plan

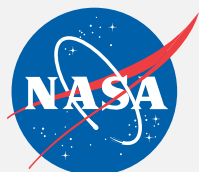
**Part E. If we want to reach  $(x_1, y_1)$ , what should the heading angle ( $\chi$ ) be?**

1. How could we solve this problem?

Have students brainstorm or work out examples independently, before sharing ideas. They might use a right triangle, use trigonometry, or use information from the previous section(s).

2. We can write a general formula to find the heading angle for use in any situation. Fill in the proof below.

	Statements	Reasons	Reference No.s
1.	$v_x = v \cos \chi$ $v_y = v \sin \chi$	Given	- - - - -
2.	$(y_1 - y_0) = (v_y / v_x)(x_1 - x_0)$	Given : Point-Slope form in C	- - - - -
3.	$(y_1 - y_0) = \frac{v \sin \chi}{v \cos \chi} (x_1 - x_0)$	Substitution	(1, 2)
4.	$\sin/\cos = \tan$	Definition of Tangent	- - - - -
5.	$y_1 - y_0 = \tan \chi (x_1 - x_0)$	Substitution	(3, 4)
6.	$\tan \chi = \frac{(y_1 - y_0)}{(x_1 - x_0)}$	Division Property of Equality	(5)
7.	$\chi = \tan^{-1} \frac{(y_1 - y_0)}{(x_1 - x_0)}$	Division Property of Equality	(6)



### Activity I: Determining the Locus of a Flight Plan

3. Solve for the heading angle, below, to the nearest hundredth degree.

	$\chi$	$y_1$	$y_0$	$x_1$	$x_0$
1.	-71.57	-3	0	1	0
2.	88.83	17	10	12	5
3.	89.91	-18	6	-25	2
4.	89.43	-12	-2	5	15
5.	-89.91	25	4	-40	-10

4. Draw each of the five situations from the table in #3. Do the heading angles agree with your drawings?

*This question provides a means for students to check their work.*

5. Now answer these questions with respect to your drawings and the chart in #3.

a) There are situations in #3 which might be better explained using other angles. What would these be?

*Negative angles could be written as positive ones.*

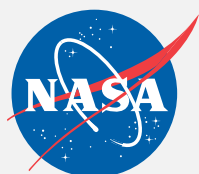
*For instance,  $-71.57^\circ = 288.43^\circ$*

*$-89.91^\circ = 270.09^\circ$*

b) If you wanted to warn a fellow student to “look out” for these situations in the future, or you wanted to generalize when you might have to change the way you write the degrees (as in Part B), what would you say? *Hint: Think about starting and ending points and their a) sign, b) relative magnitude, c) graphs you drew.*

*Be aware of situations where x decreases but y does not, or y decreases but x does not. Students might also mention directions (down and to right, for example).*

*More advanced students will enjoy the challenge of solving for other variables (as opposed to just angles). You can have them copy the previous chart and pick new values, but this time leaving other variables blank. Students can swap charts with others and get practice to use  $\tan$ ,  $\tan^{-1}$ , and working with negative numbers. Students could also use radical distances, so students could get practice adding and multiplying with radicals.*



### Activity I: Determining the Locus of a Flight Plan

#### Part F - If I want to reach $(x_1, y_1)$ at $t = t_f$ , what should I do?

This, of course, is one of the most popular questions in travel! Can you think of situations where this has been important?

You may need to have students help you rephrase the question – if I want to reach the cinema in 30 minutes, how fast do I have to go?

1. If we would like to calculate time, given specific speeds and distances, we will need to alter a few equations that we already know. Fill in the simple proof, below.

	Statements	Reasons	Reference No.s
1.	$d = (v)(t)$	Given : distance	- - - - -
2.	$d = [(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2}$	Distance Formula	- - - - -
3.	$(v)(t) = [(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2}$	Substitution	(1, 2)
4.	$t = \frac{[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2}}{v}$	Division Property of Equality	(3)

Because we are only interested in  $t_f$ , our equation is:

$$t_f = \frac{[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2}}{v}$$

2. From this equation, we can see that time is a function of

\_\_\_\_\_ original position  $(x_0, y_0)$  \_\_\_\_\_

\_\_\_\_\_ destination position  $(x_1, y_1)$  \_\_\_\_\_

and \_\_\_\_\_ velocity  $(v)$  \_\_\_\_\_ .



### Activity I: Determining the Locus of a Flight Plan

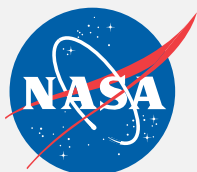
3. Fill in the following chart, solving for  $t_f$ .

Be sure to specify to students if they are to round or leave answers in exact form (radicals or fractions). Answers below are given in decimal form (to nearest hundredth) and simplified radical form.

	Where I Was	Destination	Velocity	Final Time
1.	(4, 3)	(12, 15)	4	3.61 or $(13)^{1/2}$
2.	(5, -7)	(-18, 21)	7	5.18 or $(1313^{1/2}) / 7$
3.	(-2, 9)	(-11, 17)	-3	-5
4.	(-6, -8)	(16, -10)	-4 $(2)^{1/2}$	3.91 or $(61^{1/2}) / 2$

4. What does it mean to have a negative final time? In what kind of situations does this occur? Can you think of an example? How would this situation look, graphically?

You can use the last 2 examples above, to explain. Students should show that they understand that regardless of the x and y values used, final time is negative only if the velocity is negative. Velocity is negative in situations where one is moving backwards (like the wind pushing you backwards, or the situation with a snail moving up 1 unit, then back 2 units). Realistically, these situations occur very rarely, because whoever is initiating the force will change course or wait until adverse conditions have passed, before travelling. We would call #3 and 4 in the chart "no solution" problems because there is no such thing as negative time (although some students may present travel across time zones as an example of arriving BEFORE one left).



### Activity I: Determining the Locus of a Flight Plan

5. For the previous equation (from the proof), the known variables are  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $t_f$ . If we want to solve for  $v$ , we should rewrite our equation. In other words, we will need to choose or change a specific velocity in order to reach  $(x_1, y_1)$  in a specific time period.

a) Our new equation is:

$$v = \frac{[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2}}{t}$$

b) Let's apply this notion to a trip to grandmother's house, which is in Mulino, Oregon, at latitude and longitude point (123, 45). In the following chart, fill in the following required speeds in order to get to her house from the following initial locations, with the given final times ( $t_f$ ).

Also determine what the starting points are, using a globe!

	$y_0$	$x_0$	$t_f$	Starting Point (approx)	$v$
1.	45	107	5	Sheridan, Wyoming	3.2
2.	20	160	$(997)^{1/2}$	Hawaiian Islands	1.41 or $(2)^{1/2}$
3.	37	122	$(13)^{1/2}$	Santa Cruz, California	2.24 or $(5)^{1/2}$
4.	43	0	$10(37)^{1/2}$	Sao Luis (Maranhao), off the coast of Brazil	2.02 or $(409)^{1/2} \div 10$
5.	52	130	14	Queen Charlotte Sound, off Cape St. James, Aristazabal Is.	.71 or $(2)^{1/2} \div 2$



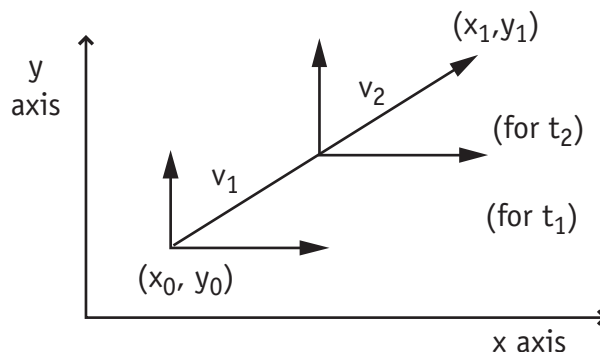
## Activity I: Determining the Locus of a Flight Plan

**Part G - If the airplane flies for a certain speed for a certain amount of time, at what speed should the airplane fly for the remaining portion of time, in order to still reach its destination at time  $t_f$ ?**

The calculations from the preceding sections are certainly useful to pilots, for determining how fast they should attempt to travel, in order to get somewhere in a specified amount of time. However, because flying speeds are usually predetermined, it is more likely that final time will be adjusted so that an airplane is not travelling at dangerously fast or slow speeds.

A more realistic situation would be if an airplane had to adjust its speed in order to reach a specific place in a certain amount of time, because weather or controller instructions had forced it to slow down for a period of time.

A picture helps explain this situation and all of the contributing factors.:



1. The total distance traveled is:

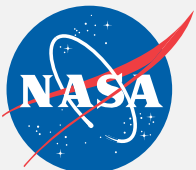
$$d = \sqrt{[(x_1 - x_0)^2 + (y_1 - y_0)^2]}^{1/2}$$

2. The distance already flown ( $d_1$ ) is:

$$d_1 = v_1 t_1$$

3. The distance remaining ( $d_2$ ) is the total distance minus the distance already flown.

$$d_2 = \sqrt{[(x_1 - x_0)^2 + (y_1 - y_0)^2]}^{1/2} - v_1 t_1$$



### Activity I: Determining the Locus of a Flight Plan

4. The remaining time ( $t_2$ ) is the final time minus the time already flown.

$$t_2 = \underline{t_f} - \underline{t_1}$$

5. We know that  $v = d/t$ .

So  $v_2 = \text{distance remaining} / \text{remaining time}$

$$\text{or } v_2 = d_2 / t_2$$

Rewrite this equation using known terms only; do not use  $t_2$  and  $d_2$ .

$$v_2 = \frac{[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2} - v_1 t_1}{t_f - t_1}$$

6. Multiply both sides of the equation by the denominator so that there are no more fractions to deal with!

$$v_2 (t_f - t_1) = [(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2} - v_1 t_1$$

7. Move extra terms to the left so you can solve for distance (in terms of x and y values).

$$v_2 (t_f - t_1) + v_1 t_1 = [(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2}$$

8. We know that

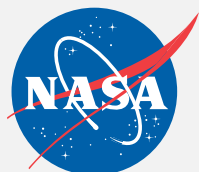
$$[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2} = vt_f$$

so

$$v_2 (t_f - t_1) + v_1 t_1 = vt_f$$

9. If we divide both sides by  $t_f$ , we can solve for velocity.

$$v = \frac{[v_2 (t_f - t_1) + v_1 t_1]}{vt_f}$$



### Activity I: Determining the Locus of a Flight Plan

10. If we attempt to make the denominator as similar to the numerator as possible, we get

$$v = \frac{v_2(t_f - t_1) + v_1 t_1}{(t_f - t_1) + t_1}$$

or

$$v = v_2 t_2 + v_1 t_1 / t_f$$

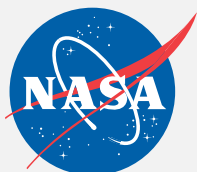
or

average speed = distance of one part + distance of other part / total time

This means that you will reach  $(x_1, y_1)$  at time  $t_f$ , as long as your average speed is  $v$ .

Using this information, fill in the following chart:

	$t_1$	$v_1$	$t_f$	$v_2$	$v$
1.	5	1	10	2	1.5
2.	5	1	20	2	1.75
3.	5	3	17	5	4.41
4.	2	7	10	8	7.8
5	1	12	15	6	6.4



### Activity I: Determining the Locus of a Flight Plan

#### Part H - Staying on the geometric path

As soon as a flight path is decided upon, the pilot and controllers monitor the course of the airplane to be sure the airplane stays on that course. This is similar to making sure a car stays on a road (and between the lines).

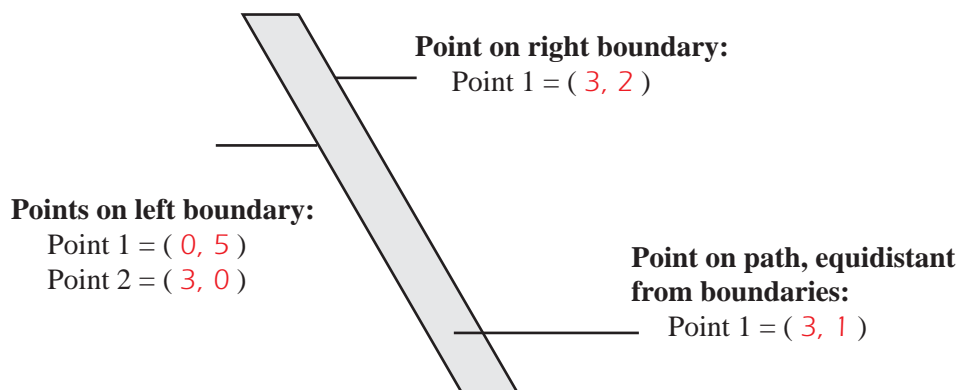
If we describe a flight path in terms of a locus defined by lines with equations, it should be easy to determine if we are inside the locus and if the actual airplane's path intersects any of the locus' boundaries (or lines).

Controllers have programs that do essentially this, and they can foresee when an airplane may leave this locus. To better understand how such programs work, let's look at a flight path specifically in terms of objective points, then regions (paths).

Let us assume that the following are paths that an airplane must adhere to. Use a piece of thin graph paper to trace the images (being sure to think of the easiest way to trace - hint: (0,0)). Pick the required points on each path and write the specified number of equations for the path.

1. For a straight path with parallel boundaries, determine the equations necessary to define the left and right boundaries, and the ideal path taken by the airplane (equidistant from the boundaries of the path).

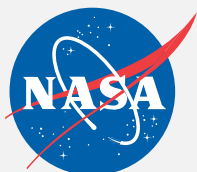
The points given here are examples only, the points chosen by the students will vary. Equations for the boundaries will vary depending on how correctly image is traced and the scale of units on the graph paper.



Equation for left boundary:  $y = -5/3 x + 5$

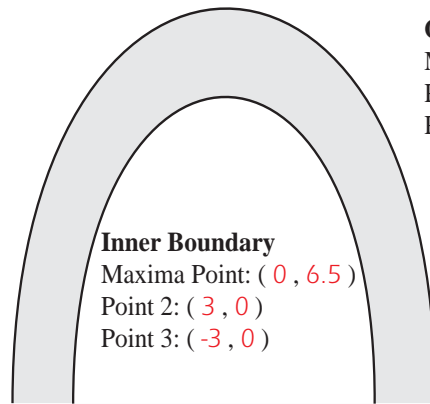
Equation for right boundary:  $y = -5/3 x + 7$

Equation for path:  $y = -5/3 x + 6$



## Activity 1: Determining the Locus of a Flight Plan

2



**Inner Boundary**  
Maxima Point: ( 0 , 6.5 )  
Point 2: ( 3 , 0 )  
Point 3: ( -3 , 0 )

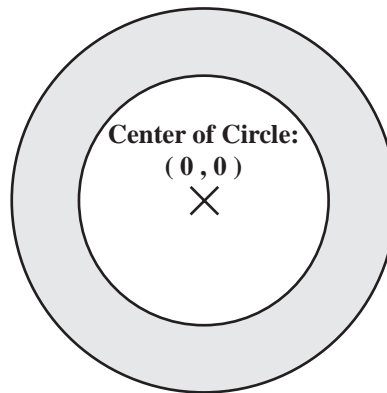
**Outer Boundary**  
Maxima Point: (      )  
Point 2: (      )  
Point 3: (      )

**On Path**  
Maxima Point: (      )  
Point 2: (      )  
Point 3: ( -3.5 , 0 )

Equation for outer boundary:  $y = -1/2 x^2 + 8$   
Equation for inner boundary:  $y = -13/18 x^2 + 13/2$   
Equation for path:  $y = -28/49 x^2 + 7$

Note that the parabolas are not perfectly shaped; the path is not equidistant from both parabolas, because the path is wider at the maxima than at the sides.

3.



**Center of Circle:**  
( 0 , 0 )  
X

**Outer Circle**  
Point 1: ( 0 , 24.5 )

**On Path**  
Point 1: ( 0 , 20.5 )

**Inner Circle**  
Point 1: ( 0 , 16.5 )

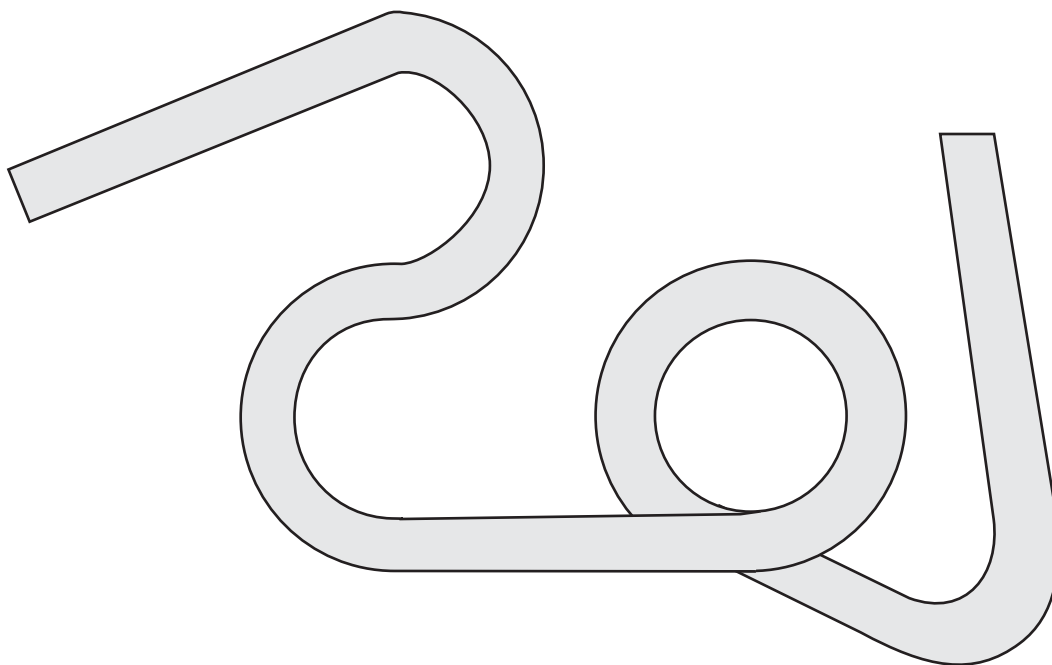
Equation for outer circle:  $24.5^2 = x^2 + y^2$   
Equation for inner circle:  $20.5^2 = x^2 + y^2$   
Equation for path:  $16.5^2 = x^2 + y^2$

This kind of activity can be easily applied to other shapes. You can give students extra practice by having them retrace the path on a new portion of the graph paper, so all of the points and equations change, although relative to each other they are all the same.



### Activity I: Determining the Locus of a Flight Plan

4. Use what you have practiced to describe mathematically this path.  
Hint: Trace the path onto graph paper, then determine the equations for each component. You may re-set coordinates or use one common coordinate system. You may also draw each component separately on different grids, to determine equations.



*Note: Equations will vary, depending on student approach.*

5. Instead of simply describing the complex path in terms of equations of lines, parabolas, or circles, it is important to include information about distances. If we combine the slope information, intercept information, and distance, we are providing the same information as that which is found in a vector.

So let's use vectors!

Use vectors to describe the path above. For complex figures like circles and parabolas, break the "rounded" figures up into multiple "straight" figures. Your teacher may give you instructions on how close your "straight" model is to the curved one - be sure to ask!

*Answers will vary, depending on how shapes are broken up.*

